Journal of Sound and Vibration (2000) **232**(3), 619–643
doi:10.1006/jsvi.1999.2768, available online at http://www.idealibrary.com on IDEAL®

NON-LINEAR OSCILLATIONS OF A BAFFLED ELASTIC PLATE IN HEAVY FLUID LOADING CONDITIONS

S. V. SOROKIN

Department of Engineering Mechanics, State Marine Technical University of St. Petersburg, Lotsmanskaya str., #3, St. Petersburg, 190008, Russia

(*Received* 26 *June* 1998, *and in final form* 3 *November* 1999)

Non-linear vibrations of an elastic plate in heavy fluid loading conditions are considered. The structural non-linearity is taken into account along with the fluid non-linearity in a full Bernoulli integral formulation for the contact acoustic pressure and with the non-linearity in the formulation of the continuity condition. The modal analysis in spatial co-ordinates is used along with the method of multiple scales to search for a stationary response in the time domain. Resonant frequencies of a fluid-loaded plate are detected in a coupled formulation of structural acoustics and typical excitation conditions (a weak resonant excitation, hard monochromatic sub- and super-harmonic excitations, an excitation by two driving forces) are explored. The roles of the structural non-linearity and the non-linearity in the formulation of the fluid response are compared.

1. INTRODUCTION

A problem of non-linear vibrations of acoustically loaded elastic structures has been thoroughly analysed by several authors, see for example, references [1–3] and literature surveys in these publications. The goal of their analyses was the inspection into possibilities of energy exchanges between vibrations at different frequencies and different modes that are uncoupled in a linear theory. It was shown by Dowell [1] that these interaction effects are produced by structural non-linearities, while an acoustical part of the problem may be formulated as a linear one in light fluid loading conditions. Moreover, in such a case relevant, say, for vibrations of structures in air, it was assumed that resonant frequencies of an elastic structure are not affected by the added mass of a surrounding acoustic medium. In the above assumptions, a theory of non-linear structural–acoustic coupling was elaborated by Abrahams [2] and Engineer and Abrahams [3] which detailed the analysis of non-linear vibrations of a baffled plate and a cylindrical shell in various excitation conditions. Special reference was made to a scattered acoustic field that in the case of a simple harmonic incident wave is enhanced by harmonics of other frequencies.

In the most general case of a non-linear acoustic medium coupled with a non-linear elastic structure Ginsberg [4] and Nayfeh and Kelly [5] have shown that the motion of the fluid at a certain distance from a vibrating body ceases to be of an acoustic nature as shock waves are developed. However, the non-linear formulation of structural–acoustic coupling in heavy fluid loading conditions is not necessarily associated with transforming the acoustic waves into the shock ones. A possibility to take into account a quadratic term in the Bernoulli integral for inspecting effects of second order in the linear acoustic field was indicated by Lord Rayleigh [6]. In the way of this concept, a recent publication [7] has

0022-460X/00/180619 + 25 $35.00/0

suggested a model of heavy loading of a non-linear elastic structure by a dense and weakly compressible fluid and several non-linear effects generated by the fluid non-linearity have been demonstrated.

As is well known, non-linear effects manifest themselves at the resonant frequencies and these excitation conditions are just the case, when various second order terms in equations of dynamics may play an important role. It is also clear, that the correct detection of resonant frequencies of a fluid-loaded structure is the essential pre-requisite for further non-linear analysis and, in the case of heavy fluid loading, such a linear problem cannot be posed for a structure vibrating in a vacuum. Thus, the aim of the present paper is to extend the non-linear structural-acoustic coupling formulation suggested in reference [7] to the case of heavy fluid loading of a baffled structure. In this case, a linear formulation of the problem includes both the added mass effect and the radiation damping effect produced by acoustic medium. The emphasis is put on the analysis of the non-linear effects in the vibration of the structure rather than in the acoustic field. Typical excitation conditions are considered with correctly found resonant frequencies of linear vibrations and a comparison of the roles of "fluid" and "structural" non-linearities is made.

A classic model problem of a baffled plate thoroughly analysed by many authors [2, 4, 5, 8–12] is taken as a case-study example. In section 2, a linear problem is briefly tackled and the resonant frequencies are detected. Section 3 contains the formulation of a multiple scales method for this problem and a solution for the problem to the order zero. In section 4, weak excitation at a resonant frequency is analysed while section 5 contains solutions for sub- and super-harmonic excitation conditions. Finally, in section 6 an example of the combinatory resonant excitation is considered.

2. LINEAR PROBLEM

As discussed in the Introduction, non-linear effects manifest themselves in resonant excitation conditions. Thus, before having a look at the non-linear fluid–structure interaction in heavy fluid loading conditions, it is essential to examine a linear formulation of the problem. The aim of re-visiting such a rather simple case is to detect correctly resonant frequencies of structural vibrations in a fluid.

Consider a model problem of linear vibration of a fluid-loaded baffled plate in a simple plane formulation relevant to cylindrical bending. Motions of a plate are described by the following equation:

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4\tilde{w}}{\partial\tilde{x}^4}+\rho_p h\frac{\partial^2\tilde{w}}{\partial t^2}+p-q=0. \tag{1}$$

Here E, ρ_p, ν and h are Young's modulus, density, the Poisson ratio and thickness of the plate, $\tilde{w}$ is the lateral displacement, q is a driving load and p is a contact acoustic pressure. An acoustic medium occupies the upper half-space and therefore the pressure acts in the opposite direction to the lateral displacement. Bending boundary conditions are imposed at $\tilde{x}=0$ and l, l being the length of a plate.

The dynamics of an acoustic medium are described by a linear wave equation formulated for the velocity potential function Φ:

$$\frac{\partial^2\Phi}{\partial\tilde{x}^2}+\frac{\partial^2\Phi}{\partial\tilde{z}^2}-\frac{1}{c_f^2}\frac{\partial^2\Phi}{\partial t^2}=0. \tag{2}$$

Here c_f is the of velocity sound in a fluid.

The compatibility condition at the fluid–structure interface is formulated as

$$\frac{\partial \Phi}{\partial \tilde{z}} = \frac{\partial \tilde{w}}{\partial t}. \tag{3}$$

Here it is taken into account that the positive direction of an outward unit normal to the fluid domain is downwards $\bar{v} = -\bar{k}$ and the positive direction of a displacement is upwards.

An acoustic pressure is defined as

$$p = -\rho_f \frac{\partial \Phi}{\partial t} \tag{4}$$

where ρ_f is the fluid density.

As is well known [6], in the case of a baffled plate linear problem in stationary acoustics, the contact pressure may be conveniently re-formulated with the use of the Rayleigh integral as

$$p(\tilde{x}, 0) = \frac{\mathrm{i}\rho_f \omega^2}{2} \int_0^l \mathrm{H}_0^{(1)} \left(\frac{\omega |\tilde{x} - \tilde{\xi}|}{c_f} \right) \tilde{w}(\tilde{\xi})\, \mathrm{d}\tilde{\xi}. \tag{5}$$

The time dependence is selected as $\exp(-\mathrm{i}\omega t)$ for a stationary solution and this multiplier is omitted in equation (5). If time dependence is specified as $\exp(+\mathrm{i}\omega \mathrm{t})$, then the Hankel function of the first kind in equation (5) should be replaced by the Hankel function of the second kind.

The substitution of equation (5) into the equation of structural dynamics and separation of time dependence gives the following non-dimensional equation for linear forced vibrations of a fluid-loaded plate:

$$w''''(x) - \mu^4 w(x) - \frac{\mathrm{i}}{2} \frac{\rho_f}{\rho_p} \frac{l}{h} \mu^4 \int_0^1 \mathrm{H}_0^{(1)} (\kappa \mu^2 |x - \zeta|)\, \mathrm{d}\zeta = \frac{12 q (1 - v^2) l^3}{E h^3}. \tag{6}$$

Here the following notations are introduced:

$$w = \frac{\tilde{w}}{l}, \quad x = \frac{\tilde{x}}{l}, \quad \mu^4 = 12(1 - v^2) \left(\frac{\omega l}{c_p} \right)^2 \left(\frac{l}{h} \right)^2, \quad \kappa = \frac{1}{\sqrt{12(1 - v^2)}} \frac{h}{l} \frac{c_p}{c_f}, \quad c_p = \sqrt{\frac{E}{\rho_p}}.$$

In the absence of a driving force ($q = 0$), one obtains the eigenvalue problem. There is a set of complex-valued eigenfrequencies that is explained by energy losses due to the radiation damping. In the literature (for example, in reference [12]), the eigenfrequencies of a fluid-loaded structure are identified with the resonant frequencies, referred to as those purely real frequencies at which in the case of forced vibrations radiated acoustic power reaches its maximum. Apparently, such a definition is justified when the real part of a complex eigenfrequency is very close to a resonant one. This is the case, for example, in the theory of structural vibrations for mechanical systems with internal and/or external damping, where a similar definition of the natural frequencies is widely used. The further study concerns resonant excitation of a plate.

Various techniques may be used for calculating resonant (or natural in the above-mentioned sense) frequencies of a fluid-loaded plate (Galerkin method, two-level boundary integral equations method, finite element method coupled with boundary element method, etc.) and we do not discuss numerical aspects of solving equation (6) here. In Figure 1, typical frequency-response curves are plotted for the plate with $h/l = 0{\cdot}01$ in the vicinity of the second resonant frequency for a simply supported plate, i.e., plate having the boundary conditions

$$w(x) = w''(x) = 0 \quad \text{at } x = 0 \text{ and } x = 1. \tag{7}$$

A plate is loaded by water so that $\rho_f/\rho_p = 0{\cdot}128$ and $c_f/c_p = 0{\cdot}308$. An amplitude, the imaginary and the real parts of a displacement at the quarter length from the edge of a plate are shown versus an excitation frequency in Figure 1(a, b, c) respectively. The amplitude of the non-dimensional driving load is chosen as $q_0/E = 0{\cdot}1835 \times 10^{-6}$, the distribution of which is in the shape of $\sin 2\pi x$. It is a standard behaviour of any damped linear mechanical system. One should note that the resonant frequency of a fluid-loaded plate in this case is

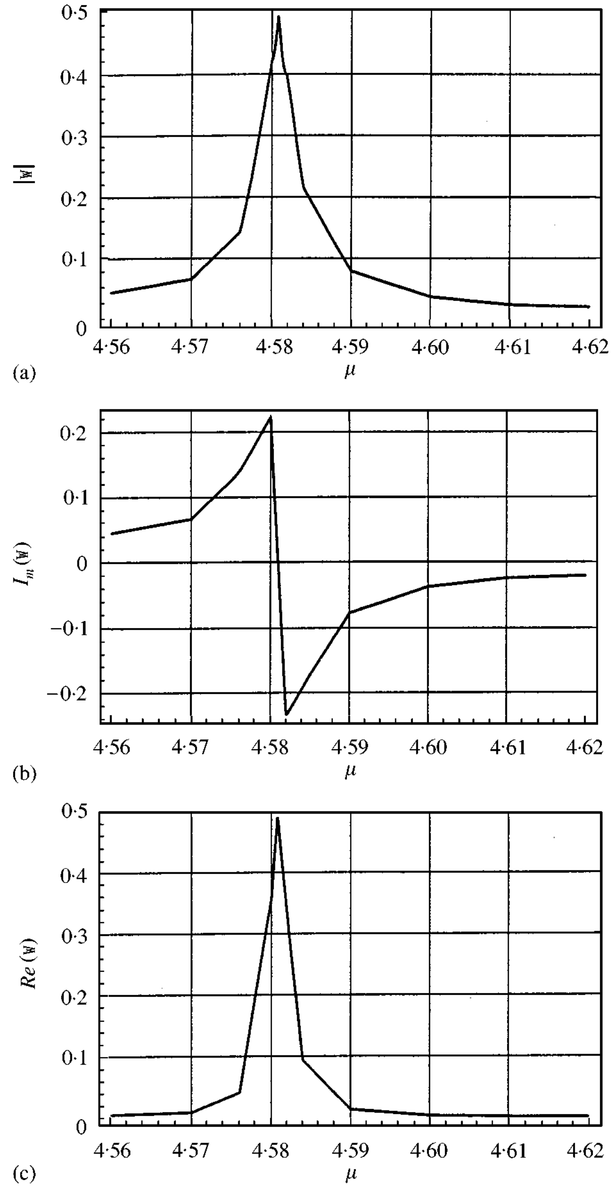

Figure 1. Frequency response near the second resonant frequency of a fluid-loaded plate having $h/l = 0{\cdot}01$. (a) Module, (b) imaginary part, (c) real part of the amplitude of displacement at quarter-span.

equal to $\mu_2 = 4{\cdot}581$ while an isolated structure has the second natural frequency of $\mu_2^0 = 2\pi$, i.e., 1·37 times higher. This case is really relevant to heavy fluid-loading conditions with the significant added mass of a fluid involved in the motion in the vicinity of the plate.

3. NON-LINEAR PROBLEM

Consider the standard non-linear formulation of the dynamics of a plate given by the equation (see e.g., Vol'mir [13] or Dowell [1])

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 \tilde{w}}{\partial \tilde{x}^4} - \frac{Eh}{2l}\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}\int_0^l \left(\frac{\partial \tilde{w}}{\partial \tilde{x}}\right)^2 \mathrm{d}\tilde{x} + \rho_p h \frac{\partial^2 \tilde{w}}{\partial t^2} + p - q = 0. \tag{8}$$

This equation in addition to the fluid-loading term contains the non-linear "stretching" term generated by the immobility of the edges of a plate in the axial direction.

Following reference [6] we assume a linear wave equation (2) to be valid in an acoustic domain with the pressure defined by the Bernoulli relation

$$p = -\rho_f \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial \tilde{z}}\right)^2 + \frac{1}{2}\left(\frac{\partial \Phi}{\partial \tilde{x}}\right)^2\right]. \tag{9}$$

As it has been shown in reference [7], the use of the full Bernoulli relation for the acoustic pressure acting on the structure makes it necessary to revise also the continuity condition which in the linear problem formulation is posed at the non-deformed fluid–structure interface; see equation (3). Instead, it should be formulated with deformation of the plate taken into account, that is [4, 7]

$$\frac{\partial \Phi}{\partial \tilde{z}} = \frac{\partial \tilde{w}}{\partial t} + \frac{\partial \tilde{w}}{\partial \tilde{x}}\frac{\partial \Phi}{\partial \tilde{x}}, \quad \tilde{z} = 0. \tag{10}$$

Equations (2, 7–10) constitute a non-linear formulation of the problem of dynamics for a fluid-loaded simply supported plate. The above formulation of the non-linear problem in structural acoustics has been suggested in reference [7]. It differs from the ones considered by other authors in the following points (i) no assumption of light fluid loading is adopted, (ii) non-linearity in the pressure formulation is treated as having the same order as the structural non-linearity and (iii) a linear wave equation describes the fluid motion in a volume.

The examination of the non-linear fluid–structure interaction will be done within a classic theory of local non-linear dynamics by the method of multiple scales applied to the time variable, see, for example references [14, 15]. This method allows the solution of equations (2), (7), (8), (9b) to be a function of independent time variables (scales). Thus, if t is written as

$$T_0 = t, \quad T_1 = \varepsilon t, \quad \varepsilon \to 0,$$

then the time derivative becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1}$$

with ε introduced as a bookkeeper of asymptotically small terms.

As is well known in the theory of non-linear oscillations, the dynamics of mechanical systems is dominated by linear restoring/inertial forces at excitation conditions which are not very close to resonance, while damping forces and the non-linearity manifest themselves at the near-resonant frequencies of excitation. Thus, in the analysis of non-linear vibrations of mechanical systems with damping, it is typical [14, 15] to assume that non-linear and damping forces produce effects of the order ε^1 ($\varepsilon \ll 1$), whereas the contribution of linear restoring and inertial forces is estimated to be of order ε^0. As we address vibrations of a fluid-loaded structure, this aspect should be discussed with some detail.

In the case of a light fluid loading, both the added mass effect (which is associated with the real part of convolution in equation (6)) and the radiation damping effect (relevant to the imaginary part of this convolution) are treated as equally small. Therefore, in the theory of non-linear dynamics of lightly fluid-loaded structures the problem of order ε^0 is posed for a structure without fluid loading [2]. As it naturally emerges from this formulation, resonant frequencies (at which the non-linear effects are to be inspected) are found as for an isolated structure. Apparently, this approach is appropriate for, say, vibrations of a steel plate in air, when indeed the added mass is very small just like the radiation damping is. However, for heavily fluid-loaded structures (as illustrated in section 2), the resonant frequencies are markedly different from those of an isolated structure.

This aspect does not reveal itself in the case of an infinitely long plate periodically supported by the immobile hinges [7] because in such a case fluid produces either the pure radiation damping effect or the pure added mass effect [12]. However, where the baffled plate is concerned, ordering terms involved in the full non-linear formulation (2), (7–10) become more sophisticated. As is shown by the simple analysis of linear vibrations of a fluid-loaded plate, the added mass plays an important role at any excitation frequency and this effect of heavy fluid loading cannot be treated as small at any frequency. On the other hand, similarly to the case of light fluid loading, the radiation damping manifests itself most pronouncedly at the resonant frequencies when in fact it produces energy outflow from a structure. It is clearly seen in Figure 1 that as the excitation frequency deviates from the resonant one, the imaginary part of an amplitude of displacement rapidly decays and, hence, so does the contribution of the radiation damping.

The above considerations permit one to assume that in the case of a heavy fluid loading of a non-linear plate the added mass effect produced by the contact acoustic pressure should be treated as a term of order ε^0. However, its radiation damping effect along with structural non-linearity should be considered in terms of order ε^1 for a rather broad class of the coupled problems. The remaining part of the present section contains derivation of equations to the first order in this assumption. Subsequent parts of the paper are concerned with an analysis of the second order equations and an estimation of the validity range of this assumption based on comparison of the asymptotic and the numerical results.

It is convenient to exclude the contact acoustic pressure from the problem formulation by substitution of Bernoulli relation (9) into equation (8):

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 \tilde{w}}{\partial \tilde{x}^4} - \frac{Eh}{2l}\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}\int_0^l \left(\frac{\partial \tilde{w}}{\partial \tilde{x}}\right)^2 \mathrm{d}\tilde{x} + \rho_p h \frac{\partial^2 \tilde{w}}{\partial t^2} - \rho_f \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial \tilde{z}}\right)^2 + \frac{1}{2}\left(\frac{\partial \Phi}{\partial \tilde{x}}\right)^2\right] - q = 0. \tag{11}$$

The analysis is restricted by the non-linear stationary formulation of a problem, so that motion of a fluid loaded plate is treated as periodic, but not necessarily monochromatic (there may be a multi-frequency motion). The principal goal of the present non-linear analysis is an inspection into possibilities of an existence of stable multi-frequency regimes of vibrations excited by a simple monochromatic loading. A set of these frequencies ϖ_j is specified in the course of analysis in each particular case. Thus, at such a regime, the wave equation is reduced to a set of Helmholtz equations at each individual frequency ϖ (the subscript j is hereafter omitted),

$$\Delta\Phi + \left(\frac{\varpi}{c_f}\right)^2 \Phi = 0.$$

Then the linear part of boundary condition (10) for acoustic domain is also split into a set of conditions formulated as

$$\frac{\partial \Phi}{\partial \tilde{z}} = \Psi(x)\exp(\mathrm{i}\,\varpi t) + \bar{\Psi}(x)\exp(-\mathrm{i}\,\varpi t), \quad \tilde{z} = 0 \tag{12}$$

(here $\Psi(x)$ is a function which is specified in the course of the analysis).

The plate is considered as being put into an infinitely long rigid baffle and the solution for a linear Helmholtz equation with the linear boundary condition (12) at $\tilde{z} = 0$ is formulated by the Rayleigh integral

$$\Phi(\tilde{x}, \tilde{z}, t)|_{\tilde{z}=0} = \frac{\mathrm{i}}{2}\left[-\int_0^l \mathrm{H}_0^{(2)}\left(\frac{\varpi|\tilde{x}-\tilde{\xi}|}{c_f}\right)\Psi(\tilde{\xi})\,\mathrm{d}\tilde{\xi}\exp(\mathrm{i}\,\varpi t) + \int_0^l \mathrm{H}_0^{(1)}\left(\frac{\varpi|\tilde{x}-\tilde{\xi}|}{c_f}\right)\bar{\Psi}(\tilde{\xi})\,d\tilde{\xi}\exp(-\mathrm{i}\,\varpi t)\right]. \tag{13}$$

Now it is convenient to perform the ordering of terms in equations (10), (11) and (13). Specifically, in equation (13) the real and the imaginary parts of the Hankel functions (relevant to the added mass and the radiation damping effects respectively) are arranged as

$$\Phi(\tilde{x}, \tilde{z}, t)|_{\tilde{z}=0} = \frac{\mathrm{i}}{2}\left\{-\int_0^l \left[\varepsilon J_0\left(\frac{\varpi|\tilde{x}-\tilde{\xi}|}{c_f}\right) - \mathrm{i}Y_0\left(\frac{\varpi|\tilde{x}-\tilde{\xi}|}{c_f}\right)\right]\Psi(\tilde{\xi})\,\mathrm{d}\tilde{\xi}\exp(\mathrm{i}\,\varpi t) + \int_0^l \left[\varepsilon J_0\left(\frac{\varpi|\tilde{x}-\tilde{\xi}|}{c_f}\right) + \mathrm{i}Y_0\left(\frac{\varpi|\tilde{x}-\tilde{\xi}|}{c_f}\right)\right]\bar{\Psi}(\tilde{\xi})\,\mathrm{d}\tilde{\xi}\exp(-\mathrm{i}\,\varpi t)\right\}. \tag{14}$$

The ordering of terms in the continuity condition (10) is

$$\frac{\partial \Phi}{\partial \tilde{z}} = \frac{\partial \tilde{w}}{\partial t} + \varepsilon\,\frac{\partial \tilde{w}}{\partial \tilde{x}}\frac{\partial \Phi}{\partial \tilde{x}}. \tag{15}$$

Finally, equation (11) is re-written as

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 \tilde{w}}{\partial \tilde{x}^4} - \varepsilon\,\frac{Eh}{2l}\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}\int_0^l \left(\frac{\partial \tilde{w}}{\partial \tilde{x}}\right)^2 \mathrm{d}x + \rho_p h\,\frac{\partial^2 \tilde{w}}{\partial t^2} - \rho_f\left[\frac{\partial \Phi}{\partial t} + \frac{1}{2}\varepsilon\left(\frac{\partial \Phi}{\partial \tilde{z}}\right)^2 + \frac{1}{2}\varepsilon\left(\frac{\partial \Phi}{\partial \tilde{x}}\right)^2\right] = [\delta_E\varepsilon + (1-\delta_E)]\,q. \tag{16}$$

Here the symbol δ_E is used to distinguish between the hard excitation ($\delta_E = 0$) and the weak excitation ($\delta_E = 1$).

An asymptotic solution can be expressed in the form

$$\tilde{w}(\tilde{x}, t) = \tilde{w}_0(\tilde{x}, T_0, T_1) + \varepsilon\tilde{w}_1(\tilde{x}, T_0, T_1), \tag{17a}$$

$$\Phi(\tilde{x}, \tilde{z}, t) = \Phi_0(\tilde{x}, \tilde{z}, T_0, T_1) + \varepsilon\Phi_1(\tilde{x}, \tilde{z}, T_0, T_1) \tag{17b}$$

The fast scale describes oscillations in "real time", while the slow one accounts for slow modulations of amplitudes and phases.

The structural problem to the order ε^0 is then formulated for the case of weak excitation at primary resonance as

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 \tilde{w}_0}{\partial \tilde{x}^4}+\rho_p h\frac{\partial^2 \tilde{w}_0}{\partial T_0^2}-\rho_f\frac{\partial \Phi_0}{\partial T_0}=0. \tag{18a}$$

In the case of a hard sub- or super-harmonic excitation it is

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 \tilde{w}_0}{\partial \tilde{x}^4}+\rho_p h\frac{\partial^2 \tilde{w}_0}{\partial T_0^2}-\rho_f\frac{\partial \Phi_0}{\partial T_0}=q. \tag{18b}$$

The Rayleigh integral (14) to the order ε^0 is formulated as

$$\Phi_0(\tilde{x},\tilde{z},T_0)|_{\tilde{z}=0}=-\frac{1}{2}\left\{\int_0^l \mathrm{Y}_0\left(\frac{\varpi|\tilde{x}-\tilde{\xi}|}{c_f}\right)\Psi(\tilde{\xi})\,\mathrm{d}\tilde{\xi}\exp(\mathrm{i}\varpi T_0)\right.$$
$$\left.+\int_0^l \mathrm{Y}_0\left(\frac{\varpi|\tilde{x}-\tilde{\xi}|}{c_f}\right)\bar{\Psi}(\tilde{\xi})\,\mathrm{d}\tilde{\xi}\exp(-\mathrm{i}\varpi T_0)\right\}.$$

As already discussed, the essential effect of a heavy fluid loading is the shift of resonant frequencies of a plate from those of an isolated one. To find these resonant frequencies, a homogeneous problem (18a) should be solved. The shape of the vibrations of the structure is sought in the form

$$w_0=A_0X_k(\tilde{x})\exp(\mathrm{i}\omega_{0k}T_0)+A_0X_k(\tilde{x})\exp(-\mathrm{i}\omega_{0k}T_0), \tag{19}$$

where ω_{0k} is a resonant frequency and $X_k(x)$ is the kth resonant mode of vibrations. In equation (19), both the resonant frequency and the resonant mode of vibrations should be found, whereas A_0 is an undetermined amplitude independent of time in the standard linear analysis.

The continuity condition (15) to the order ε^0 is the same as in the linear theory,

$$\frac{\partial \Phi_0}{\partial \tilde{z}}=\frac{\partial \tilde{w}_0}{\partial T_0}=\mathrm{i}\omega_k A_0X_k(\tilde{x})\exp(\mathrm{i}\omega_{0k}T_0)-\mathrm{i}\omega_k A_0X_k(\tilde{x})\exp(-i\omega_{0k}T_0). \tag{20}$$

Comparison of equations (12) and (20) leads to the conclusion that $\varpi=\omega_{0k}$ and $\Psi=-\bar{\Psi}=\mathrm{i}\omega_{0k}A_0X_k(\tilde{x})$. Thus, the velocity potential at the surface of a plate becomes

$$\Phi_0(\tilde{x},\tilde{z},T_0)|_{\tilde{z}=0}=\frac{i\omega_{0k}}{2}\left\{-\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right)X_k(\tilde{\xi})\,\mathrm{d}\tilde{\xi}A_0\exp(\mathrm{i}\omega_{0k}T_0)\right.$$
$$\left.+\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right)X_k(\tilde{\xi})\,\mathrm{d}\tilde{\xi}A_0\exp(-\mathrm{i}\omega_{0k}T_0)\right\}. \tag{21}$$

When equations (19) and (21) are inserted into the equation of structural dynamics (18a), the following problem is formulated

$$X_k''''(x)-\mu_k^4X_k(x)+\frac{1}{2}\frac{\rho_f}{\rho_p}\frac{l}{h}\mu_k^4\int_0^1 \mathrm{Y}_0(\kappa\mu_k^2|x-\zeta|)X_k(\zeta)\,\mathrm{d}\zeta=0,\quad \mu_k^4=12(1-\nu^2)\left(\frac{\omega_{0k}l}{c_p}\right)^2\left(\frac{l}{h}\right)^2 \tag{22}$$

with the boundary conditions (7). Here non-dimensional quantities $x=\tilde{x}/l$, $\xi=\tilde{\xi}/l$ are used, and the amplitude A_0 and time dependence $\exp(\pm\mathrm{i}\omega_{0k}t)$ are omitted.

Hence, in this approximation, the resonant frequencies and resonant modes of vibrations of a plate in heavy fluid loading conditions are to be found as the eigenmodes and eigenfrequencies of the boundary eigenvalue problem (7), (22). It is appropriate to discuss briefly the physical interpretation of such a problem. Apparently, if the limit case of an incompressible fluid is considered, $\mu_k \to 0$, then the Bessel function $Y_0(\kappa\mu_k^2|x-\xi|)$ is expanded into a series and the first term is retained. Then, in effect, a problem of free vibrations of a baffled plate in an incompressible fluid is posed. In this low-frequency limit, the incompressible fluid produces a pure added mass effect and eigenvalues of the problem (22), (7) are purely real. Such a problem formulation might be of some interest, but its validity is strictly limited by the assumption $\mu_k \to 0$, i.e., by inspection into quasi-static cases. The formulation (22) is also real-valued and it also gives only added mass effect similar to the incompressible case. However, it is free from the assumption $\mu_k \to 0$ and the added mass encountered in equation (22) consists of both the added mass of an incompressible fluid and the added mass of a compressible fluid presented by subsequent terms in the expansion of the Bessel function $Y_0(\kappa\mu_k^2|x-\xi|)$. In some cases, the 'acoustic' part added mass may substantially contribute to its overall value, as it has been shown, for example, in reference [16] for a cylindrical shell.

Before studying the non-linear effects in the vibrations of a baffled plate, it is necessary to justify the validity of the assumption concerning separation of real and imaginary parts in the problem to order ε^0. One of the justifications is provided by a direct comparison of the eigenvalues of the problem (22), (7) with the results of the linear analysis performed in Section 2.

Eigenmodes and eigenvalues of the problem (22), (7) may be found by various methods, see for example references [10, 11]. The simplest one is the use of the Galerkin procedure. Then eigenmodes $X_k(x)$ are expanded into a set of orthogonal trial functions satisfying boundary conditions for a plate and equation (22) is orthogonalized to each trial function. It is convenient to select a set of trial functions as eigenmodes of an isolated plate (with no fluid loading) having the same boundary conditions as the fluid-loaded one. In fact, it does not present any difficulties to proceed further with no specification of boundary conditions. However, in order to have an estimation of the roles of the structural non-linearity and the fluid non-linearity it is sufficient to deal with the most convenient case of a simply supported plate. Undoubtedly, in other boundary conditions somewhat different quantitative results may be obtained, but qualitatively the roles of the above effects should remain the same.

Thus, the boundary conditions are selected as those given by equation (7) and the eigenmodes of the boundary eigenvalue problem (22) are expanded as

$$X_k(x) = \sum_m B_{km} \sin m\pi x. \tag{23}$$

To facilitate convergence of the expansion (23) it is convenient to use the spatial symmetry of the problem formulation with respect to the centre of the plate. There is a set of symmetric modes which is uncoupled with a set of skew-symmetric ones. This means, that for the odd numbers k only odd harmonics should be retained in equation (23) while for the even numbers k only even harmonics should be taken.

The Galerkin procedure results in the following system of linear algebraic equations:

$$\sum_m B_{km}\left[\frac{1}{2}\delta_{km}((m\pi)^4-\mu_k^4)+\frac{1}{2}\frac{\rho_f}{\rho_p}\frac{l}{h}\mu^4\int_0^1\int_0^1 Y_0(\kappa\mu_k^2|x-\zeta|)\sin k\pi x\sin m\pi\zeta\,\mathrm{d}\zeta\,\mathrm{d}x\right]=0. \tag{24}$$

The doubled integral in square brackets by some elementary manipulations is transformed as

$$\int_0^1\int_0^1 \mathrm{Y}_0\,(\kappa\mu_k^2\,|\,x-\zeta\,|)\sin k\pi x\sin m\pi\zeta\,\mathrm{d}\zeta\,\mathrm{d}x = (1+(-1)^{m+k})\int_0^1 C_k(x)\sin k\pi x\sin m\pi x\,\mathrm{d}x$$
$$-(1-(-1)^{m+k})\int_0^1 S_k(x)\cos k\pi x\sin m\pi x\,\mathrm{d}x, \tag{25}$$

where

$$C_k(x)=\int_0^x \mathrm{Y}_0\,(\kappa\mu_k^2\,\zeta)\cos k\pi\zeta\,\mathrm{d}\zeta$$

and

$$S_k(x)=\int_0^x \mathrm{Y}_0\,(\kappa\mu_k^2\,\zeta)\sin k\pi\zeta\,\mathrm{d}\zeta.$$

As seen from equation (25), the second term is identically equal to zero if the numbers m and k are not both odd or both even. Thus, equation (24) may be re-written as

$$\sum_m B_{km}\left[\delta_{km}\,((m\pi)^4-\mu_k^4)+2\,\frac{\rho_f}{\rho_p}\frac{l}{h}\,\mu^4\int_0^1 C_k(x)\sin k\pi x\sin m\pi x\,\mathrm{d}x\right]=0. \tag{26}$$

The first three resonant frequencies of a plate of $h/l=0{\cdot}01$ in contact with water ($\rho_f/\rho_p=0{\cdot}128$, $c_f/c_p(= 0{\cdot}308)$ have been calculated and their values are presented in the first row of Table 1. These frequencies have been obtained when five odd and five even terms are retained in expansion (23).

To verify the validity of the way suggested for the determination of the resonant frequencies, a comparison between the values found in the framework of this problem formulation and values relevant to maximum amplitudes at forced vibrations (see section 1) was performed. Within the tolerance of computation no difference has been found between resonant frequencies, obtained by these two techniques, therefore no separate row is inserted in Table 1. Their values were also compared with results presented in references [10, 11] (the second row in Table 1). All these values are in good agreement. To give an idea of the contribution of the added mass effect, the third row in Table 1 contains values of natural frequencies of an isolated plate. It is clearly seen that the effect of the added mass is much pronounced. Apparently, this effect is controlled by the parameter h/l as illustrated in Figure 2, where the first two resonant frequencies of a fluid-loaded plate are shown versus the parameter $\log_{10}(l/h)$. A value of $\mu=\pi$ attributed to the first (symmetric) mode of vibration for an isolated plate with $h/l\approx 0{\cdot}002$ in the case of loading by water is related to the second (skew-symmetric) mode of a fluid-loaded plate. As one addresses the problem of the identification of resonant frequencies and structural non-linear dynamics at these frequencies such a mixing up of different modes is rather dangerous.

To conclude this part, it should be pointed out that it might be doubtful to claim the imaginary parts of the computed displacements of a fluid-loaded plate to be small as compared with their real parts in out-of-resonant excitation conditions. However, the very good agreement between resonant frequencies calculated in section 2 and eigenvalues of the problem (22), (7) justifies the asymptotic ordering of terms in equations (14–16). The radiation damping appears to be similar to the light structural damping, which does not markedly shift the resonant frequencies from the eigenfrequenies of an undamped

TABLE 1

Eigenfrequencies of a baffled plate

1·623	4·581	7·235
1·516	4·536	7·326
π	2π	3π

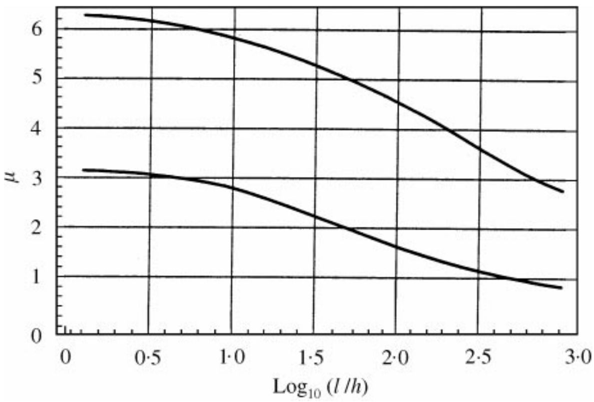

Figure 2. The first two eigenfrequencies of a fluid-loaded plate versus thickness of a plate.

mechanical system. More evidence in favour of this ranging is given in the next part of the paper, as well as an example when this assumption does not hold.

4. MODAL ANALYSIS OF WEAK EXCITATION

The examination of the non-linear effects in the vibration of a baffled plate begins with the case of a weak excitation. Physically, it means that a resonant driving force of relatively small magnitude may produce pronounced non-linear effects. Formally, a driving force is multiplied by ε.

In this case a solution to the problem of order ε^0 is sought in the form

$$\tilde{w}_0(\tilde{x}, T_0, T_1) = A_0(T_1) X_k(\tilde{x}) \exp(\mathrm{i}\omega_{0k} T_0) + \bar{A}_0(T_1) X_k(\tilde{x}) \exp(-\mathrm{i}\omega_{0k} T_0), \qquad (27)$$

i.e., a single-mode analysis is performed. This equation differs from equation (19) by letting the amplitude $A_0(T_1)$ be a slow varying complex function. It should be noted here that both the eigenfrequency ω_{0k} and the eigenmode $X_k(x)$ are solutions of the eigenvalue problem (22), (7), see section 3.

A problem in structural dynamics of the order ε^1 is formulated as

$$\frac{Eh^3}{12(1-\nu^2)} \frac{\partial^4 \tilde{w}_1}{\partial \tilde{x}^4} + \rho_p h \frac{\partial^2 \tilde{w}_1}{\partial T_0^2} - \rho_f \frac{\partial \Phi_1}{\partial T_0} = q - 2 \frac{\partial^2 \tilde{w}_0}{\partial T_0 \partial T_1} + \sum_{j=1}^{4} Q_{0j}. \qquad (28)$$

Here a chain rule is applied to formulate $\partial^2 \tilde{w}/\partial t^2$ in equation (16); the driving load is specified as $q(\tilde{x}) = q_0 Q(\tilde{x}) \cos(\omega_{0k} T_0 + \sigma T_1)$ with σ induced as a detuning parameter indicating how close the driving frequency is to the resonant frequency ω_{0k}. The left-hand side of equation (28) has exactly the same form as the zeroth order equation (18a). It contains displacement $\tilde{w}_1$ and velocity potential Φ_1 formulated by the second terms of the asymptotic expansions (17a,b). For simplicity, driving load is assumed to be distributed fairly close to the shape of the kth eigenmode. The velocity potential Φ_1 is defined by the wave equation (2) and a linear part of the continuity condition (10), i.e., in the same way as Φ_0 in the problem to order ε^0.

$$\frac{\partial^2 \Phi_1}{\partial \tilde{x}^2} + \frac{\partial^2 \Phi_1}{\partial \tilde{z}^2} - \frac{1}{c_f^2}\frac{\partial^2 \Phi_1}{\partial T_0^2} = 0, \tag{29a}$$

$$\frac{\partial \Phi_1}{\partial \tilde{z}} = \frac{\partial \tilde{w}_1}{\partial T_0}. \tag{29b}$$

The right-hand side of equation (28) includes a driving load and several terms depending upon $\tilde{w}_0$. The first term taken into account as Q_{01} is related to the structural non-linearity

$$Q_{01} = \frac{Eh}{2l}\frac{\partial^2 \tilde{w}_0}{\partial \tilde{x}^2}\int_0^l \left(\frac{\partial \tilde{w}_0}{\partial \tilde{x}_1}\right)^2 \mathrm{d}\tilde{x}_1 \tag{30}$$

The second one is relevant to the "radiation damping" part of Rayleigh integral (14), i.e.,

$$Q_{02} = \frac{\mathrm{i}\rho_f \omega_{0k}^2}{2}\left\{\int_0^l \mathrm{J}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right) X_k(\tilde{\xi})\,\mathrm{d}\tilde{\xi} A_0(T_1)\exp(\mathrm{i}\omega_{0k}T_0)\right.$$
$$\left. - \int_0^l \mathrm{J}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right) X_k(\tilde{\xi})\,\mathrm{d}\tilde{\xi}\,\bar{A}_0(T_1)\exp(-\mathrm{i}\omega_{0k}T_0)\right\}. \tag{31}$$

The third term is generated by the quadratic non-linearity in the Bernoulli relation

$$Q_{03} = \frac{1}{2}\rho_f\left[\left(\frac{\partial \Phi_0}{\partial \tilde{z}}\right)^2 + \left(\frac{\partial \Phi_0}{\partial \tilde{x}}\right)^2\right]. \tag{32}$$

Its first component is formulated straightforwardly by the use of the compatibility condition to the order ε^0, i.e.,

$$\left.\frac{\partial \Phi_0}{\partial \tilde{z}}\right|_{\tilde{z}=0} = \frac{\partial \tilde{w}_0}{\partial T_0} = \mathrm{i}\omega_{0k}\left[A_0(T_1)X_k(\tilde{x})\exp(\mathrm{i}\omega_{0k}T_0) - \bar{A}_0(T_1)X_k(\tilde{x})\exp(-\mathrm{i}\omega_{0k}T_0)\right] \tag{33a}$$

The velocity potential Φ_0 at the fluid–structure interface is given by equation (21) up to terms of the order ε^0. Hence, differentiation of equation (21) gives

$$\left.\frac{\partial \Phi_0}{\partial \tilde{x}}\right|_{\tilde{z}=0} = \frac{\mathrm{i}\omega_0^2}{2c_f}\left\{\int_0^l \mathrm{Y}_1\left(\frac{\omega_0|\tilde{x}-\tilde{\xi}|}{c_f}\right)\mathrm{sign}\,(\tilde{x}-\tilde{\xi})\,X_k(\tilde{\xi})\,\mathrm{d}\tilde{\xi}A_0(T_1)\exp(\mathrm{i}\omega_0 T_0)\right.$$
$$\left. - \int_0^l \mathrm{Y}_1\left(\frac{\omega_0|\tilde{x}-\tilde{\xi}|}{c_f}\right)\mathrm{sign}\,(\tilde{x}-\tilde{\xi})\,X_k(\tilde{\xi})\,\mathrm{d}\tilde{\xi}\bar{A}_0(T_1)\exp(-\mathrm{i}\omega_0 T_0)\right\}.$$

It is convenient to use the symmetry of the Green function for a fluid

$$\frac{\partial}{\partial \tilde{x}}\mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right) = -\frac{\omega_{0k}}{c_f}\mathrm{Y}_1\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right)\mathrm{sign}\,(\tilde{x}-\tilde{\xi}) = -\frac{\partial}{\partial \tilde{\xi}}\mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right)$$

and perform integration by parts. Non-integral terms vanish due to boundary conditions (7) and $\partial \Phi_0/\partial \tilde{x}|_{\tilde{z}=0}$ becomes

$$\left.\frac{\partial \Phi_0}{\partial \tilde{x}}\right|_{\tilde{z}=0} = \frac{\mathrm{i}\omega_{0k}}{2}\left\{\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right) X_k'(\tilde{\xi})\,\mathrm{d}\tilde{\xi} A_0(T_1)\exp(\mathrm{i}\omega_{0k}T_0) - \int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right) X_k'(\tilde{\xi})\,\mathrm{d}\tilde{\xi}\bar{A}_0(T_1)\exp(-\mathrm{i}\omega_{0k}T_0)\right\}. \tag{33b}$$

The fourth term on the right hand side of equation (28) is generated by the non-linearity in the continuity condition (10),

$$Q_{04} = \rho_f \frac{\partial \Phi_{01}}{\partial T_0}.$$

The component Φ_{01} of a velocity potential is defined by a wave equation and the non-linear part of continuity condition (10),

$$\frac{\partial \Phi_{01}}{\partial \tilde{z}} = \frac{\partial \tilde{w}_0}{\partial \tilde{x}} \left.\frac{\partial \Phi_0}{\partial \tilde{x}}\right|_{\tilde{z}=0} \tag{34}$$

As equations (27) and (33) are substituted into the boundary condition (34) the latter becomes

$$\frac{\partial \Phi_{01}}{\partial \tilde{z}} = [A_0(T_1)X_k'(\tilde{\xi})\exp(\mathrm{i}\omega_{0k}T_0) + \bar{A}_0(T_1)X_k'(\tilde{\xi})\exp(-i\omega_{0k}T_0)] \times\left[-\frac{i\omega_{0k}}{2}\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{\xi}-\tilde{\xi}_1|}{c_f}\right) X_k'(\tilde{\xi}_1)\,\mathrm{d}\tilde{\xi}_1 A_0(T_1)\exp(\mathrm{i}\omega_{0k}T_0) + \frac{\mathrm{i}\omega_{0k}}{2}\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{\xi}-\tilde{\xi}_1|}{c_f}\right) X_k'(\tilde{\xi}_1)\,\mathrm{d}\tilde{\xi}_1 \bar{A}_0(T_1)\exp(-\mathrm{i}\omega_{0k}T_0)\right]. \tag{35}$$

The frequency-independent term in equation (35) is of no interest because it cannot be secular in equation (29a) at any frequency. Then formula (35) is simplified to

$$\frac{\partial \Phi_{01}}{\partial \tilde{z}} = -\frac{\mathrm{i}\omega_{0k}}{2}\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{\xi}-\tilde{\xi}_1|}{c_f}\right) X_k'(\tilde{\xi}_1)\,\mathrm{d}\tilde{\xi}_1 A_0^2(T_1)X_k'(\tilde{\xi})\exp(2\mathrm{i}\omega_{0k}T_0) + \frac{\mathrm{i}\omega_{0k}}{2}\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{\xi}-\tilde{\xi}_1|}{c_f}\right) X_k'(\tilde{\xi}_1)\,\mathrm{d}\tilde{\xi}_1 \bar{A}_0^2(T_1)X_k'(\tilde{\xi})\exp(-2\mathrm{i}\omega_{0k}T_0),$$

and it is clear that a linear wave equation for Φ_{01} is transformed to the Helmholtz equation

$$\Delta \Phi_{01} + \left(\frac{2\omega}{c_f}\right)^2 \Phi_{01} = 0$$

so that the Rayleigh integral (21) is formulated as

$$\Phi_{01}(\tilde{x}) = \frac{\mathrm{i}\omega_{0k}}{4}\left\{-\int_0^l \mathrm{Y}_0\left(\frac{2\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right) X_k'(\tilde{\xi})\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{\xi}_1-\tilde{\xi}|}{c_f}\right) X_k'(\tilde{\xi}_1)\,\mathrm{d}\tilde{\xi}\,\mathrm{d}\tilde{\xi}_1 A_0^2 \times\exp(2\mathrm{i}\omega_{0k}T_0) + \int_0^l \mathrm{Y}_0\left(\frac{2\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right) X_k'(\tilde{\xi})\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{\xi}_1-\tilde{\xi}|}{c_f}\right) \times X_k'(\tilde{\xi}_1)\,\mathrm{d}\tilde{\xi}\,\mathrm{d}\tilde{\xi}_1 \bar{A}_0^2\exp(-2\mathrm{i}\omega_{0k}T_0)\right\}.$$

The fourth term on the right-hand side of equation (28) is formulated as

$$Q_{04} = \frac{\rho_f \omega_{0k}^2}{2} \left\{ \int_0^l \mathrm{Y}_0 \left(\frac{\omega_{0k} |\tilde{x} - \tilde{\xi}|}{c_f} \right) X_k'(\tilde{\xi}) \int_0^l \mathrm{Y}_0 \left(\frac{\omega_{0k} |\tilde{\xi}_1 - \tilde{\xi}|}{c_f} \right) X_k'(\tilde{\xi}_1) \,\mathrm{d}\tilde{\xi}\,\mathrm{d}\tilde{\xi}_1 A_0^2(T_1) \right.$$

$$\times \exp(2\mathrm{i}\omega_{0k} T_0) + \int_0^l \mathrm{Y}_0 \left(\frac{\omega_{0k} |\tilde{x} - \tilde{\xi}|}{c_f} \right) X_k'(\tilde{\xi}) \int_0^l \mathrm{Y}_0 \left(\frac{\omega_{k0} |\tilde{\xi}_1 - \tilde{\xi}|}{c_f} \right) X_k'(\tilde{\xi}_1) \,\mathrm{d}\tilde{\xi}\,\mathrm{d}\tilde{\xi}_1$$

$$\left. \times \bar{A}_0^2(T_1) \exp(-2\mathrm{i}\omega_{0k} T_0) \right\}. \tag{36}$$

After the Galerkin orthogonalization of equation (28) to the kth mode $X_k(\tilde{x})$, the secular terms on the right-hand side of equation (28) are presented as ($x = \tilde{x}/l$, $\xi = \tilde{\xi}/l$)

$$\left\{ \frac{6(1-\nu^2)\,l^3}{Eh^3} q_0 \exp(\mathrm{i}\sigma T_1) \int_0^1 Q(x)\, X_k(x)\,\mathrm{d}x - \frac{18(1-\nu^2)\,l^2}{Eh^2} A_0^2 \bar{A}_0 \left[\int_0^1 [X_k'(x)]^2 \,\mathrm{d}x \right]^2 \right.$$

$$+ \frac{6\mathrm{i}(1-\nu^2)\,l^5 \omega_{0k}^2}{c_p^2 h^3} \frac{\rho_f}{\rho_p} A_0 \int_0^1 \int_0^1 J_0 \left(\frac{\omega_{0k}\, l}{c_f} |x - \xi| \right) X_k(x)\, X_k(\xi)\,\mathrm{d}\xi\,\mathrm{d}x$$

$$\left. - \frac{24\mathrm{i}(1-\nu^2) l^4 \omega_{0k}}{c_p^2 h^2} \frac{\partial A_0}{\partial T_1} \int_0^1 X_k^2(x)\,\mathrm{d}x \right\} \exp(\mathrm{i}\omega_{0k} T_0) + \mathrm{cc}. \tag{37}$$

To ensure a uniform validity of the asymptotic expansion (17a), the terms in curly brackets in equation (37) should be removed. Elimination of resonant terms in equation (37) results in the following amplitude modulation equation:

$$\frac{q_0}{E} \frac{l}{h} Z_q \mathrm{e}^{\mathrm{i}\sigma T_1} - 3 Z_s A_0^2 \bar{A}_0 + \frac{l}{h} \frac{\rho_f}{\rho_p} \left(\frac{\omega_k l}{c_p} \right)^2 \mathrm{i} Z_f A_0 - \frac{4 l^2}{c_p^2} \mathrm{i}\omega_k \frac{\mathrm{d}A_0}{\mathrm{d}T_1} Z_0 = 0. \tag{38}$$

Here

$$\omega_k \equiv \omega_{0k} \quad Z_q = \int_0^1 Q(x)\, X_k(x)\,\mathrm{d}x, \quad Z_s = \left(\int_0^1 (X_k'(x))^2\,\mathrm{d}x \right)^2, \quad Z_0 = \int_0^1 X_k^2(x)\,\mathrm{d}x,$$

$$Z_f = \int_0^1 \int_0^1 \mathrm{J}_0 (\kappa\mu^2 |x - \zeta|\, X_k(\zeta)\, X_k(x)\,\mathrm{d}x\,\mathrm{d}\zeta.$$

It is worth noting that a "fluid non-linearity" (the quadratic in velocity term $\partial w_0/\partial T_0$ in equations (32) and (36)) has not produced secular terms in the modulation equation (38), i.e., non-linear dynamics in weak resonant excitation conditions is controlled only by the structural non-linearity.

A complex-valued function $A_0(T_1)$ is conveniently expressed via two real-valued functions: $A_0(T_1) = \frac{1}{2} a(T_1) \mathrm{e}^{\mathrm{i}\varphi(T_1)}$. Then equation (38) is replaced by two equations with respect to an amplitude a and a phase φ:

$$\frac{2\omega_k l^2}{c_p^2} Z_0 a \frac{\partial \varphi}{\partial T_1} = -\frac{q_0}{E} \frac{l}{h} Z_q \cos(\sigma T_1 - \varphi) + \frac{3}{8} Z_s a^3, \tag{39a}$$

$$\frac{2\omega_k l^2}{c_p^2} Z_0 \frac{\partial a}{\partial T_1} = \frac{q_0}{E} \frac{l}{h} Z_q \sin(\sigma T_1 - \varphi) + \frac{1}{2} \frac{\rho_f}{\rho_p} \frac{l}{h} \left(\frac{\omega_k l}{c_p} \right) Z_f\, a. \tag{39b}$$

It is convenient to introduice the phase as $\psi = \sigma T_1 - \varphi$. As we search for a stationary regime, we set $\partial\psi/\partial T_1 = \partial a/\partial T_1 = 0$ and obtain the following equations for an amplitude a and a phase ψ:

$$\theta^2 a^6 - 2\theta\beta\left(\frac{\Omega}{\omega_k} - 1\right)a^4 + \left[\alpha^2 + \beta^2\left(\frac{\Omega}{\omega_k} - 1\right)^2\right]a^2 - \gamma^2 = 0, \tag{40a}$$

$$\tan\psi = \frac{\alpha}{\beta(\Omega/\omega_k - 1) - \theta a^2}. \tag{40b}$$

Here

$$\alpha = \frac{1}{2}\frac{l}{h}\left(\frac{\omega_k l}{c_p}\right)^2 \frac{\rho_f}{\rho_p} Z_f, \quad \beta = 2Z_0\left(\frac{\omega_k l}{c_p}\right)^2, \quad \theta = \frac{3}{8}Z_s, \quad \gamma = \frac{q_0}{E}\frac{l}{h}Z_q.$$

The roots of the cubic in a^2, equation (40a), are obtained by the use of symbolic manipulator *Mathematica* [17], but they are very cumbersome and therefore not presented here.

Several assumptions have been adopted while deriving equations (40). Therefore, it is necessary to verify the validity of these equations before performing the analysis of the results obtained with them. Such a verification may easily be performed since the method of multiple scales is equally applicable to an analysis of vibrations of non-linear systems and to an analysis of vibrations of damped linear systems [14, 15]. If the non-linear terms produced by structural non-linearity are dropped from equation (40), then predictions from this equation may be compared with numerical solution of a linear problem, see Section 2. This is important since such a comparison should verify the validity of ordering the real part of the Rayleigh integral (14) and its imaginary part. In Figure 3, the dependence of an amplitude of vibrations upon the excitation frequency parameter μ is presented in the vicinity of the first resonant frequency of the plate with $h/l = 0{\cdot}1$. Even for such a thick plate, the influence of the surrounding fluid (water) results in a decrease of resonant natural frequency parameter of about 10%, see Figure 3(a). The amplitude of the driving force is selected as $q_0/E = 0{\cdot}1865 \times 10^{-3}$ (such a high level of loading, not feasible in practice, is chosen to provide large amplitudes of displacements of a thick plate). Curve 1 is plotted for the amplitude of vibrations at the centre of a plate obtained from equation (40a) with no structural non-linearity included, curve 2 gives the same frequency dependence obtained from a numerical solution of a linear problem. It is seen that there is a reasonable agreement between them except in the narrow vicinity of a resonant frequency. The latter may be explained by insufficient accuracy of numerical integration and interpolation of convolution integrals (25) which shows up only in this frequency range.

The good agreement between these two curves outside this frequency range justifies the ordering of the terms in the Rayleigh integral (14). It should be pointed out that the numerical solving of linear problem is rather time-consuming and should be repeated for each frequency, whereas to perform analysis of the plate's behaviour using equations (40), it is necessary to calculate coefficients Z_s, Z_0, Z_f, Z_q only once. In Figure 3(b), curve 1 is the same as in Figure 3(a), but the frequency range is somewhat broader, while curve 2 presents the amplitude of vibration of a non-linear plate at the same excitation conditions. It is clear that resonant behaviour of a plate is controlled by structural non-linearity. However, acoustic loading influences not only the value of resonant frequency, but also the shape of frequency response curves via parameter Z_f.

Another example is relevant to vibrations of the plate with $h/l = 0{\cdot}01$ resonantly excited by the load of $q_0/E = 0{\cdot}1865 \times 10^{-6}$ in the frequency range in the vicinity of the second natural frequency $\mu_2 = 4{\cdot}581$, see Figure 4a. Curve 1 is obtained numerically through solving the linear problem. Curve 2 presents the amplitude of vibration of a plate obtained

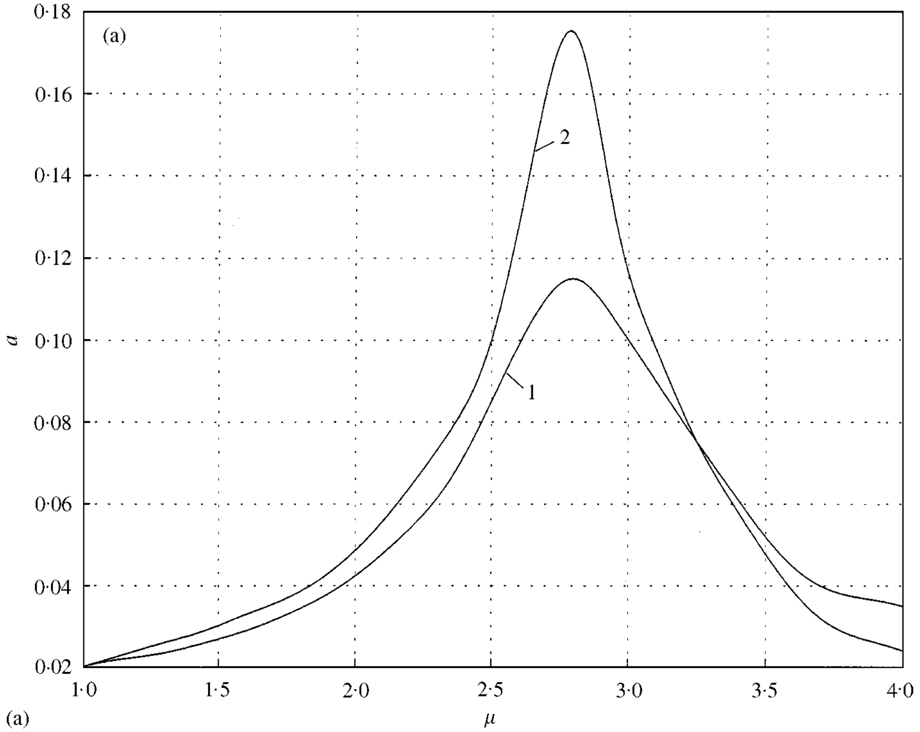

Figure 3(a). Frequency response at the first resonance of a plate with $h/l = 0{\cdot}1$. The linear theory. Comparison between numerical results (curve 2) and predictions by the method of multiple scales (curve 1).

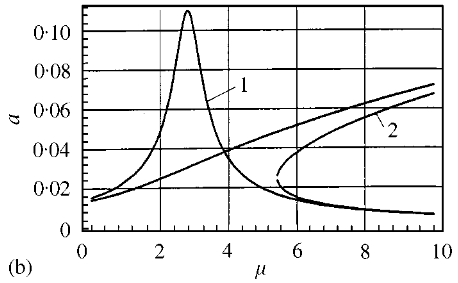

Figure 3(b). Frequency response at the first resonance of a plate with $h/l = 0{\cdot}1$. Comparison between predictions by the method of multiple scales for a linear theory (curve 1) and non-linear theory (curve 2)

by the method of multiple scales with acoustical loading, but with no structural non-linearity (equation (40a), $Z_S = 0$). Finally, curve 3 gives the amplitude of vibration of an acoustically loaded non-linear plate calculated via equation (40a), $Z_S \neq 0$. Again, a good agreement is observed between predictions by the multiple scales method for an acoustically damped linear plate (curve 2) and the direct solution of a linear problem (curve 1). Resonant phenomena in a linear formulation of a problem are displayed in a very narrow frequency band. Physically, it may be explained by the fact that a skew-symmetric mode of plate's motions provokes motions of acoustic medium very similar to those in the case of incompressible fluid. Therefore, a quality of the resonance is markedly higher in the direct numerical solution and in this case it is also higher than in the previous one. However, as

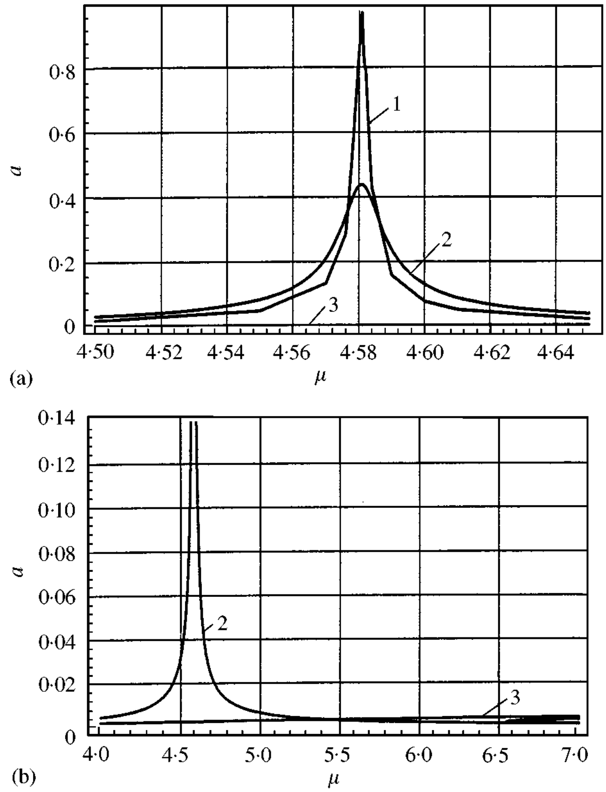

Figure 4(a). Frequency response at the second resonance of a plate with $h/l = 0{\cdot}01$. Comparison between numerical results of a linear theory (curve 1), predictions by the method of multiple scales in linear theory (curve 2), and predictions by the method of multiple scales in non-linear theory (curve 3). (b) The same as in (a) for somewhat broader frequency range.

shown by curve 3, the non-linearity completely absorbs all effects of the resonant excitation in this case, i.e., it "cuts off" the resonant peak. In Figure 4(b), the linear (curve 2) and the non-linear (curve 3) responses are plotted against each other in somewhat broader frequency range. A comparison of these two curves shows that ignoring the structural non-linearity gives a typical peak of amplitude bounded by the radiation damping, whereas the structural non-linearity once taken into account entirely controls the behaviour of a fluid-loaded plate in this case. Thus, one may conclude that in this particular case fluid loading effects are negligibly small as compared with the effect of in-plane stretching, and non-linear vibrations of a plate may be considered as uncoupled from fluid loading. The third example is relevant to the excitation of the same plate at the vicinity of the third resonant frequency, $\mu_3 = 7{\cdot}235$. The amplitude of the driving force is the same as in the previous case, but its distribution is now selected to be similar to the third eigenmode. In this case, in addition to good agreement between two solutions of a linear problem it is seen that the role of non-linearity is less pronounced, see Figure 5. This phenomenon may be explained by a balance between the acoustic damping of vibrations and the non-linear effects.

The last example concerns vibrations of a very thin plate, $h/l = 0{\cdot}003$. The modal analysis is performed in the vicinity of the first resonant frequency of a fluid-loaded plate, $\mu_1 = 1{\cdot}123$. In this case, the fluid produces a very heavy damping (as it produces a large added mass)

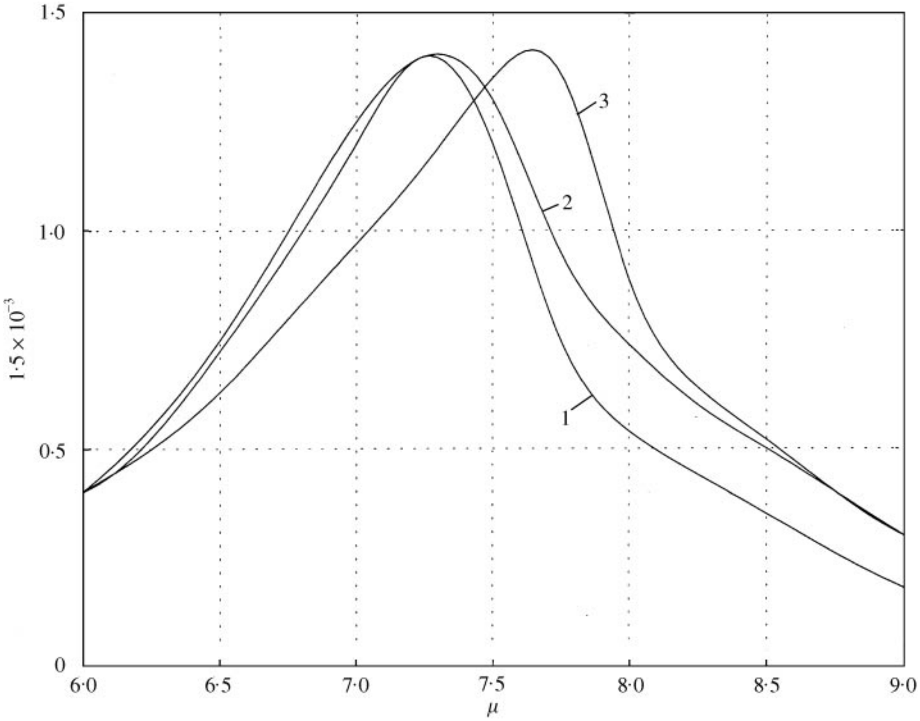

Figure 5. Frequency response at the third resonance of a plate with $h/l = 0{\cdot}01$. Comparison between numerical results of a linear theory (curve 1), predictions by the method of multiple scales in linear theory (curve 2), and predictions by the method of multiple scales in non-linear theory (curve 3).

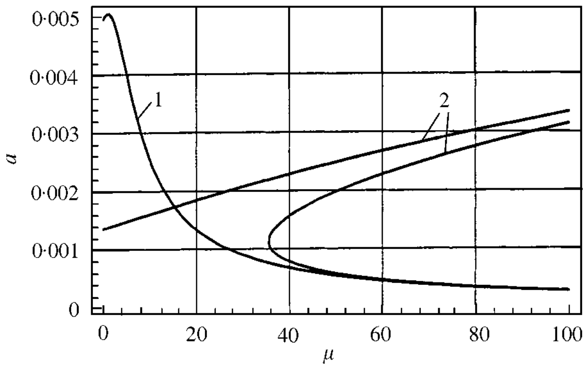

Figure 6. Frequency response at the first resonance of a plate with $h/l = 0{\cdot}003$. Comparison of a linear theory (curve 1) and a non-linear theory (curve 2).

and some revision of the asymptotic formulation of the problem is actually required. This conclusion follows from the examination of the frequency scale in Figure 6. Resonant behaviour of a linear plate subject to acoustical loading is predicted by the method of multiple scales in a by far too broad range of frequencies, see curve 1 in Figure 6, whereas the linear analysis predicts a rather sharp resonant peak, not displayed in this Figure. This disagreement naturally indicates validity limits of asymptotic ordering of terms in Rayleigh integral adopted in the analysis. However, it is also seen that a non-linear response (curve 2) is very different from the linear one and for such a thin plate structural non-linearity dominates radiation damping.

Summing up the above examples of non-linear stationary dynamics at the weak resonant excitation, it should be noted that the "fluid non-linearity" does not control the dynamics of a fluid-loaded plate in considered excitation conditions, because no secular terms are supplied to equation (37) by Q_{03} and Q_{04}. The fluid's contribution to the amplitude modulation equation (40a) is relevant only to the radiation damping, whereas the structural non-linearity is taken into account in equation (40a). This equation may be reliably used to predict the response of the non-linear plate in heavy fluid loading conditions in most cases. When a plate is rather thick, contributions of the radiation damping and structural non-linearity are of the same order and they are adequately modelled in a single-term modal analysis. When a plate is thin, its non-linearity completely absorbs fluid-loading effects and the insufficient accuracy of the description of the radiation damping does not play an important role.

5. MODAL ANALYSIS OF HARD EXCITATION

Excitation conditions when a driving frequency is close to the resonant one are of most importance from the practical viewpoint. However, there are other regimes of excitation that may result in vibrations at the resonant frequency as non-linear dynamics is considered. Here two regimes will be explored in detail: the sub-harmonic excitation ($\Omega \approx \omega_k/2$) and the super-harmonic excitation, when $\Omega \approx 2\omega_k$, $\omega_k \equiv \omega_{0k}$. The particular interest in these regimes is explained by the fact that 'fluid non-linearity' which does not enter the amplitude modulation equations in the case of a weak resonant excitation should manifest itself in both the above cases. Then it will be instructive to compare a contribution of the fluid non-linearity with a contribution of the structural one.

As the driving force is away from the near-resonant region, then a full forced response at the driving frequency may be obtained by solving a linear problem (18b), (7). In both these cases, a problem of order ε^0 should be solved as a problem of forced vibrations of a plate with fluid added mass. For simplicity assume that the driving force is distributed in the shape of the kth eigenmode, or, more precisely, consider this component of the driving force, i.e., $q(x) = q_0 X_k(x)$. Then a solution for a problem of order ε^0 is presented as

$$w_0(x,t) = \lfloor A_0(T_1)\mathrm{e}^{\mathrm{i}\omega_k T_0} + \bar{A}_0(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_k T_0} + A_q(\mathrm{e}^{\mathrm{i}R(\omega_k T_0+\sigma T_1} + \mathrm{e}^{-\mathrm{i}R(\omega_k T_0+\sigma T_1)})\rfloor X_k(x), \tag{41a}$$

$$\begin{aligned}\Phi_0(\tilde{x},\tilde{z},T_0) = \frac{\mathrm{i}\omega_{0k}}{2}\Bigg\{ &-\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right) X_k(\tilde{\xi})\,\mathrm{d}\tilde{\xi} A_0(T_1)\exp(\mathrm{i}\omega_{0k}T_0)\\ &+\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right) X_k(\tilde{\xi})\,\mathrm{d}\tilde{\xi}\bar{A}_0(T_1)\exp(-\mathrm{i}\omega_{0k}T_0)\Bigg\}\\ &+\frac{\mathrm{i}R\omega_{0k}}{2}\Bigg\{-\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right) X_k(\tilde{\xi})\,\mathrm{d}\tilde{\xi} A_q\exp(\mathrm{i}\mathrm{R}\omega_{0k}T_0 + \mathrm{i}\sigma T_1)\\ &+\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{c_f}\right) X_k(\tilde{\xi})\,\mathrm{d}\tilde{\xi} A_q\exp(-\mathrm{i}R\omega_{0k}T_0 - i\sigma T_1)\Bigg\}.\end{aligned} \tag{41b}$$

Here $R = \frac{1}{2}$ or 2 in the cases of sub- and super-harmonic excitations, respectively.

The amplitude of forced vibrations is found from the equation

$$A_q \int_0^1 \left[X_k^{(4)}(x) - \mu_R^4 X_k(x) + \frac{1}{2}\frac{\rho_f}{\rho_p}\frac{l}{h}\mu_R^4 \int_0^1 \mathrm{Y}_0(\kappa\mu_R^2 |x-\xi|) X_k(\xi)\mathrm{d}\xi \right] X_k(x)\,\mathrm{d}x$$

$$= \frac{12(1-\nu^2)\,l^3}{Eh^3} q \int_0^1 X_k^2(x)\,\mathrm{d}x, \quad \mu_R^4 = 12(1-\nu^2)\left(\frac{R\omega_k l}{c_f}\right)^2 \left(\frac{l}{h}\right)^2. \tag{42}$$

To determine the amplitude of vibrations at the resonant frequency ω_k it is necessary to solve the problem to order ε^1:

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 \tilde{w}_1}{\partial \tilde{x}^4} + \rho_p h \frac{\partial^2 \tilde{w}_1}{\partial T_0^2} - \rho_f \frac{\partial \Phi_1}{\partial T_0} = -2\frac{\partial^2 \tilde{w}_0}{\partial T_0 \partial T_1} + \sum_{j=1}^{4} Q_{0j}. \tag{43}$$

Now two cases should be distinguished, i.e., $R = \frac{1}{2}$ and 2. In both these cases, four terms on the right-hand side of equation (43) are defined exactly as in the case of weak excitation. Namely, the formulae (41) should be substituted into equations (30–32) and (36).

Consider a case of sub-harmonic excitation, i.e., $R = \frac{1}{2}$. Then non-linear fluid loading resonant terms in equation (43) are formulated as

$$Q_{03} = -\left\{ \frac{1}{8}\rho_f \omega_{0k}^2 X_k^2(\tilde{x}) + \frac{1}{32}\rho_f \omega_{0k}^2 \left[\int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{2c_f}\right) X'(\tilde{\xi})\,\mathrm{d}\tilde{\xi} \right]^2 \right\}$$

$$\times [\exp(\mathrm{i}\omega_{0k}T_0 + \mathrm{i}\sigma T_1) + \exp(-\mathrm{i}\omega_{0k}T_0 - \mathrm{i}\sigma T_1)]\, A_q^2,$$

$$Q_{04} = -\frac{1}{8}\rho_f \omega_{0k}^2 \int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{x}-\tilde{\xi}|}{2c_f}\right) X'(\tilde{\xi}) \int_0^l \mathrm{Y}_0\left(\frac{\omega_{0k}|\tilde{\xi}-\tilde{\xi}_1|}{2c_f}\right) X'(\tilde{\xi}_1)\,\mathrm{d}\tilde{\xi}_1\,\mathrm{d}\tilde{\xi}$$

$$\times [\exp(\mathrm{i}\omega_{0k}T_0 + \mathrm{i}\sigma T_1) + \exp(-\mathrm{i}\omega_{0k}T_0 - \mathrm{i}\sigma T_1)]\, A_q^2.$$

In these excitation conditions, Q_{01} does not contain secular terms, whereas the secular terms in Q_{02} are given by equation (31). The condition for the absence of secular terms in equation (43) is

$$-Z_s(3A_0^2\bar{A}_0 + 6A_0A_q^2) + \left(\frac{\omega_k l}{c_p}\right)^2 \frac{l}{h}\frac{\rho_f}{\rho_p}\mathrm{i}A_0 Z_f - \frac{1}{4}\left(\frac{\omega_k l}{c_p}\right)^2 \frac{l}{h}\frac{\rho_f}{\rho_p} A_q^2\,\mathrm{e}^{\mathrm{i}\sigma T_1} Z_n$$

$$-\frac{4l^2\omega_k}{c_p^2}\, i\, \frac{\partial A_0}{\partial T_1} Z_0 = 0. \tag{44}$$

Here, in addition to the notation, introduced earlier,

$$Z_n = \int_0^1 X_k^3(x)\,\mathrm{d}x + \frac{1}{4}\int_0^1 X_k(x) \left[\int_0^1 \mathrm{Y}_0\left(\frac{\omega_{0k}|x-\xi|}{2c_f}\right) X_k'(\xi)\,\mathrm{d}\xi \right]^2 \mathrm{d}x$$

$$+ \int_0^1 X_k(x) \int_0^1 \mathrm{Y}_0\left(\frac{\omega_{0k}|x-\xi|}{2c_f}\right) X_k'(\xi) \int_0^1 \mathrm{Y}_0\left(\frac{\omega_{0k}|\xi-\xi_1|}{2c_f}\right) X_k'(\xi_1)\,\mathrm{d}\xi_1\,\mathrm{d}\xi\,\mathrm{d}x$$

A complex-valued function $A_0(T_1)$ is conveniently expressed via two real-valued functions: $A_0(T_1) = \frac{1}{2}a(T_1)\,\mathrm{e}^{\mathrm{i}\varphi(T_1)}$. Then equation (44) is replaced by two equations with respect to an amplitude a and a phase φ. Similar to the case of weak excitation, a phase is introduced as

$\psi = \sigma T_1 - \varphi$. Then to identify a stationary regime, the conditions $\partial\psi/\partial T_1 = \partial a/\partial T_1 = 0$ should hold, and the following equations for an amplitude a and a phase ψ are obtained:

$$\theta^2 a^6 - 2\theta\left[\left(\frac{\Omega}{\omega_k} - \frac{1}{2}\right)\beta - \vartheta\right]a^4 + \left[\alpha^2 + \left[\beta\left(\frac{\Omega}{\omega_k} - \frac{1}{2}\right) - \vartheta\right]^2\right]a^2 - \chi^2 = 0, \quad (45a)$$

$$\tan\psi = \frac{\alpha}{\beta(\Omega/\omega_k - \frac{1}{2}) - \theta a^2 - \vartheta}. \quad (45b)$$

Here, in addition to the notation introduced earlier, $\vartheta = 3Z_s A_q^2$ and $\chi = \frac{1}{4} Z_n (\omega_k l/c_p)^2 \times (l/h)(\rho_f/\rho_p) A_q^2$. It should be pointed out that the possibility of sub-harmonic excitation is determined only by "fluid" non-linearity. As the quadratic in the velocity term is dropped, we have $\chi = 0$ and one can see that a zero solution for equation (45) is stable. Further examination shows that the existence of the solution at the resonant frequency is dependent upon a value of Z_n. If this quantity is close to zero, then an unrealistically large forced amplitude A_q (and, hence, a very large excitation force) at the driving frequency is required for the energy transfer into resonant motions. For any odd mode, Z_n is very small indeed. The situation is slightly different if an even mode is considered. For example, in the case of sub-harmonic hard excitation of a plate with $h/l = 0{\cdot}01$ at its third natural frequency, the amplitude of resonant motions may reach up to 0·5–1% of the amplitude at the driving frequency. This level of excitation is in good agreement with the results of similar analysis performed for an infinitely long plate supported by a set of equally spaced immobile hinges [7]. Therefore, one may conclude that despite the fact that the "fluid" non-linearity produces a qualitatively new effect of excitation of resonant motions at $\Omega \approx \omega_k/2$, quantitatively this effect is almost negligible.

In the case of super-harmonic excitation ($R = 2$), to remove secular terms from the right-hand side of the equation of order ε^1 the following condition should hold:

$$- Z_s(3A_0^2\bar{A}_0 + 6A_0A_q^2) + \mathrm{i}\left(\frac{\omega_{0k} l}{c_p}\right)^2 \frac{l}{h}\frac{\rho_f}{\rho_p} Z_f A_0 + 4\left(\frac{\omega_{0k} l}{c_p}\right)^2 \frac{l}{h}\frac{\rho_f}{\rho_p} Z_n \bar{A}_0 A_q \exp(2\mathrm{i}\sigma T_1)$$
$$- 4\mathrm{i}Z_0\left(\frac{\omega_{0k} l}{c_p}\right)^2 \frac{\partial A_0}{\omega_{0k}\partial T_1} = 0. \quad (46)$$

Here

$$Z_n = \int_0^1 X_k^3(x)\,\mathrm{d}x + \frac{1}{4}\int_0^1 X_k(x)\int_0^1 \mathrm{Y}_0\left(\frac{\omega_{0k}|x-\xi|}{c_f}\right)X_k'(\xi)\,\mathrm{d}\xi \int_0^1 \mathrm{Y}_0\left(\frac{2\omega_{0k}|x-\xi_1|}{c_f}\right)X_k'(\xi_1)\,\mathrm{d}\xi_1\,\mathrm{d}x$$
$$+ \frac{1}{16}\int_0^1 X_k(x)\int_0^1 \mathrm{Y}_0\left(\frac{\omega_{0k}|x-\xi|}{c_f}\right)X_k'(\xi)\int_0^1 \mathrm{Y}_0\left(\frac{\omega_{0k}|\xi-\xi_1|}{c_f}\right)X_k'(\xi_1)\,\mathrm{d}\xi_1\,\mathrm{d}\xi\,\mathrm{d}x$$
$$- \frac{1}{4}\int_0^1 X_k(x)\int_0^1 \mathrm{Y}_0\left(\frac{\omega_{0k}|x-\xi|}{c_f}\right)X_k'(\xi)\int_0^1 \mathrm{Y}_0\left(\frac{2\omega_{0k}|\xi-\xi_1|}{c_f}\right)X_k'(\xi_1)\,\mathrm{d}\xi\,\mathrm{d}\xi_1\,\mathrm{d}x.$$

Standard transformations discussed in the previous cases result in the following amplitude modulation equations:

$$- \alpha a^3 - \delta a + \beta a\left(\frac{\Omega}{\omega_k} - 2\right) = a\eta\cos\psi \quad (47a)$$

$$\alpha a = a\eta\sin\psi. \quad (47b)$$

Here, in addition to the notation introduced earlier, $\eta = 2Z_n(\omega_k l/c_p)^2 (l/h)(\rho_f/\rho_p) A_q$.

This system of equations has a trivial solution $a = 0$ that is relevant to the absence of resonant motions of a plate in super-harmonic excitation conditions. Non-trivial solutions are easily found from the quadratic equation in a^2 and given by

$$a^2 = \frac{1}{\theta}\left[\beta\left(\frac{\Omega}{\omega_k} - 2\right) - \delta \mp \sqrt{\eta^2 - \alpha^2}\right]. \tag{48}$$

Apparently, positive values of a^2 may be obtained from equation (48) if the following inequality holds:

$$A_q \geqslant Z_f/4Z_n\,. \tag{49}$$

One may easily check that this condition may be fulfilled only if a forced amplitude A_q is unrealistically large not only for even, but also for odd modes of vibrations. For example, in the earlier considered case of a plate with $h/l = 0{\cdot}01$ vibrating in its first mode, the right-hand side of equation (49) is equal to 0·0985, i.e., forced vibrations should have a magnitude of almost 10 times larger than the thickness. If the third mode is considered, then this number is 0·0528. Although the right-hand side is decreasing with growth of frequency number, it should be noted that from a practical viewpoint, super-harmonic effects are of no interest.

Summing up the results of this section, it should be pointed out that the effects produced by the fluid non-linearity are very weak both in the sub- and the super-harmonic excitation conditions. Therefore, assumptions concerning the roles of the structural non-linearity and the fluid non-linearity formulated in reference [1] for the light fluid-loading conditions, are also held in the case of heavy fluid-loading conditions. Nevertheless, the difference between the heavy fluid-loading theory and the light fluid-loading theory (a necessity to take into account the fluid added mass effect in the evaluation of resonant frequencies) emphasized in the present paper is of principal importance in detecting conditions for non-linear effects of any origin to be exposed.

6. COMBINATORY HARD RESONANT EXCITATION

Finally, the case of a combinatory resonant excitation is briefly tackled. This kind of hard excitation is relevant to the situation when two driving loads at different frequencies Ω_1 and Ω_2 act on the plate and the following condition holds: $\Omega_1 + \Omega_2 \approx \omega_{0k}$ (ω_{0k} is the natural frequency of a plate loaded by an acoustic medium).

In this case, a solution for the problem of order ε^0 is sought as

$$\begin{aligned} w_0(x,t) = {} & [A_0(T_1)\mathrm{e}^{\mathrm{i}\omega_k T_0} + \bar{A}_0(T_1)\mathrm{e}^{-\mathrm{i}\omega_k T_0} + A_{q13}(\mathrm{e}^{(1/3)\mathrm{i}(\omega_k T_0 + \sigma T_1)} + \mathrm{e}^{-(1/3)\mathrm{i}(\omega_k T_0 + \sigma T_1)})]X_k(x) \\ & + A_{q23}\,(\mathrm{e}^{(2/3)\mathrm{i}(\omega_k T_0 + \sigma T_1)} + \mathrm{e}^{-(2/3)\mathrm{i}(\omega_k T_0 + \sigma T_1)})\,X_k\,(x). \end{aligned} \tag{50}$$

The amplitudes of forced vibrations A_{q13} and A_{q23} at each of two excitation frequencies are found from equation (42) with $R = \frac{1}{3}$ and $\frac{2}{3}$ respectively. The left-hand side of equation (43) contains secular terms that will be removed if the following condition holds:

$$\begin{aligned} & -Z_s\left(\frac{3}{8}a^3 + 3a(A_{q13}^2 + A_{q23}^2)\right) - Z_s\,(A_{13}^3 + 3A_{q13}\,A_{q23}^2)\exp\mathrm{i}\psi + \frac{\mathrm{i}}{2}\left(\frac{\omega_k l}{c_p}\right)^2 \frac{l}{h}\frac{\rho_f}{\rho_p} Z_F\, a \\ & -\frac{8}{9}\left(\frac{\omega_k l}{c_p}\right)^2 \frac{l}{h}\frac{\rho_f}{\rho_p} Z_n A_{13}A_{23}\exp\mathrm{i}\psi - \frac{2l^2\omega_k Z_0}{c_p^2}\,\mathrm{i}\,\frac{\partial a}{\partial T_1} \\ & + \frac{2l^2\omega_k Z_0}{c_p^2}\,a\left[\omega_k\left(\frac{\Omega}{\omega_k} - \frac{1}{3}\right) - \frac{\partial \psi}{\partial T_1}\right] = 0. \end{aligned} \tag{51}$$

In equation (51), a phase is introduced as $\psi = \sigma T_1 - \varphi$. Then to identify a stationary regime, conditions $\partial\psi/\partial T_1 = \partial a/\partial T_1 = 0$ are imposed and the following equations for amplitude a and phase ψ are obtained:

$$\left[\left(\beta\left(\frac{\Omega}{\omega_k} - \frac{1}{3}\right) - \upsilon\right) a - \theta a^3\right]^2 + (\alpha a)^2 = \chi^2, \tag{52a}$$

$$\tan\psi = \frac{\alpha}{\beta(\Omega/\omega_k - \frac{1}{3}) - \theta a^2 - \upsilon}. \tag{52b}$$

Here $\chi = Z_s(A_{q13}^3 + 3A_{q13}A_{q23}^2) + \frac{8}{9}(\omega_k l/c_p)^2 (l/h)(\rho_f/\rho_p) Z_n A_{q13} A_{q23}$ and $\vartheta = 3Z_s \times (A_{q13}^2 + A_{q23}^2)$.

Modulation equations in the case of a monochromatic excitation at the frequency $\frac{1}{3}\omega_{0k}$ are easily obtained from equations (50–52) by putting the amplitude A_{q23} to zero. Thus, excitation at the frequency $\frac{2}{3}\omega_{0k}$ just adds three terms to the formulae for χ and υ relevant to the monochromatic excitation at $\frac{1}{3}\omega_{0k}$. Nevertheless, this modification produces the qualitatively new effect that is illustrated in Figure 7(a, b). Curve 1 in Figure 7(a) displays the resonant response at the hard monochromatic sub-harmonic excitation at $\Omega = \frac{1}{3}\omega_3$ of a plate having $h/l = 0{\cdot}01$. The amplitude of displacement at the mid-span is scaled to the amplitude of displacement at the excitation frequency. As clearly seen from this

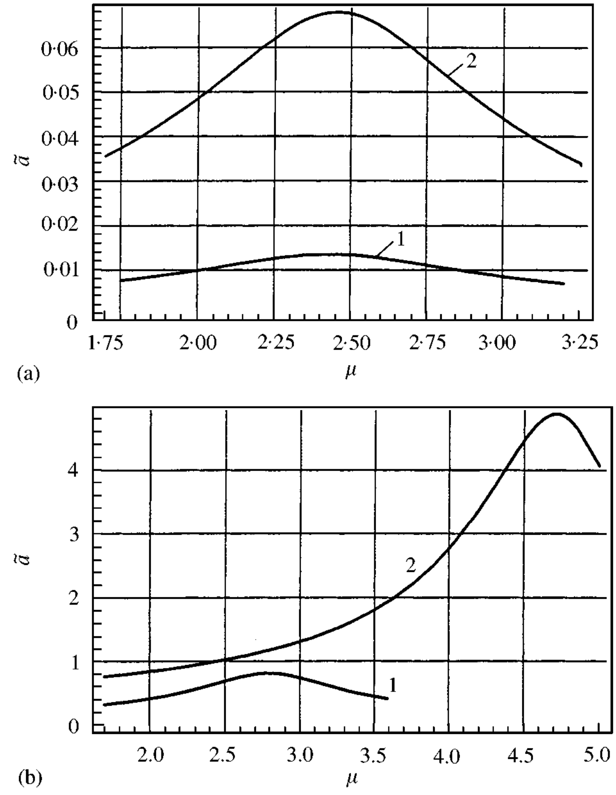

Figure 7(a). Resonant response at combinatory excitation conditions for a plate with $h/l = 0{\cdot}01$. The lower curve: $q_{23} = 0$, the upper curve: $q_{23} = q_{13}$. (b) Resonant response at combinatory excitation conditions for a plate with $h/l = 0{\cdot}01$. The lower curve: $q_{23} = 4q_{13}$, the upper curve: $q_{23} = 10q_{13}$.

graph, the resonant response is quite weak. Curve 2 in Figure 7(a) gives the same response at the combinatory excitation when the amplitude of the driving force at frequency $\frac{2}{3}\,\omega_3$ is equal to the amplitude of the driving force at $\frac{1}{3}\,\omega_3 : q_{23} = q_{13}$. The resonant response is still rather weak, but it is markedly higher than in the previous case. In Figure 7(b), curves 1 and 2 present the same resonant response for the cases $q_{23} = 4q_{13}$ and $q_{23} = 10q_{13}$ respectively. In addition to an amplification of the resonant amplitude, there is also a shift of its maximum to the frequency of $\frac{2}{3}\omega_3$. The amplitude is still scaled to the amplitude of the forced response at $\frac{1}{3}\,\omega_3$, but re-scaling to the amplitude of the response at $\frac{2}{3}\,\omega_3$ shows that resonant motions become of the same magnitude as the motions at the forced frequency. This simple example demonstrates that the excitation at the frequency of $\frac{2}{3}\,\omega_{0k}$ may provoke a large resonant response. One should note that a monochromatic excitation at this frequency alone cannot produce any resonant effects. However, the presence of the small component at “complementary” frequency $\frac{1}{3}\,\omega_{0k}$ triggers rather strong resonant motions. Apparently, one should expect similar effects at other combinations of driving frequencies when the condition $\Omega_1 + \Omega_2 \approx \omega_{0k}$ holds. In the case considered in this section, both the structural non-linearity and the fluid non-linearity in principle contribute to the amplitude modulation equation, but similar to the cases treated in sections 4 and 5, the structural non-linearity dominates the fluid one.

7. CONCLUSIONS

An investigation has been completed into non-linear vibrations of elastic plates in heavy fluid-loading conditions. It is shown that to detect resonant frequencies in such a case, it is necessary to formulate the problem to order ε^0 with the added mass of the fluid taken into account. Then an acoustical damping along with non-linear terms enters the problem to order ε^1. This simple result is important because natural frequencies of a plate determined in an uncoupled formulation (i.e., with no fluid loading) may be much higher than the resonant frequencies in a coupled formulation (i.e., with a fluid loading). Since the non-linear phenomena manifest themselves at excitation frequencies like, for example, a certain fraction of the resonant one, it is essential to find the latter correctly. Similarly, in the non-linear analysis performed in this paper, the resonant modes of heavily fluid-loaded plates are used rather than the ones relevant to vibrations in vacuum.

The method of multiple scales is used to consider a weak resonant excitation. To validate asymptotic results obtained by useing this method, resonant vibrations of a fluid-loaded plate are examined in a linear problem formulation. Firstly, it is found that the resonant frequencies may be reliably detected as the eigenvalues of a boundary eigenvalue problem posed with only the fluid added mass effect taken into account. Secondly, the amplitudes of forced near-resonant vibrations of an acoustically damped linear plate are compared with results of the direct linear analysis and good agreement is demonstrated, except in the narrow vicinity of sharp resonant peaks for rather thin plates. However, these are just the cases when non-linear effects play an important role. In the non-linear analysis of a weak resonant excitation, it is found that non-linear effects are controlled solely by the structural non-linearity, whereas the fluid non-linearity does not enter the amplitude modulation equations. In the case of a very thin plate (especially, at the first resonance), structural non-linearity completely absorbs fluid-loading effects and an insufficient accuracy of description of the radiation damping (which is also detected in this case) does not play any important role. Then a plate is rather thick (and at the higher resonances of a thin plate), contributions of the acoustical linear radiation damping and the structural non-linearity become of the same order and they are adequately modelled in a single-term modal analysis adopted in the paper.

To estimate the role of the fluid non-linearity, a hard sub-harmonic excitation at the frequency $\Omega \approx \omega_k/2$ is considered. It is found that resonant motions may be excited only at natural frequencies relevant to even eigenmodes, and this effect is fairly weak. Similarly, resonant motions generated by the fluid non-linearity at super-harmonic excitation $\Omega \approx 2\omega_k$ are unstable, whereas "zero solution" for resonant motions in these excitation conditions is stable. Therefore, one may conclude that fluid non-linearity does not contribute much to the non-linear dynamics of a plate having structural non-linearity in heavy fluid-loading conditions.

Finally, in the case of a combinatory hard excitation, it is found that large-amplitude resonant motions can be provoked by the interaction of small-amplitude vibrations at $\frac{1}{3}\,\omega_{0k}$ and large-amplitude vibrations at $\frac{2}{3}\,\omega_{0k}$. It should be pointed out that the latter kind of excitation does not generate resonant motions without the presence of vibrations at frequency of $\frac{1}{3}\,\omega_{0k}$. In this case, the structural non-linearity also dominates the fluid non-linearity.

REFERENCES

1. E. H. Dowell 1975 *Aeroelasticity of Plates and Shells*. Leyden: Noordhoff.
2. I. D. Abrahams 1987 *Proceedings of the Royal Society of London* **A414**, 237–253. Acoustic scattering by a finite nonlinear elastic plate, I. Primary, secondary and combination resonances.
3. J. C. Engineer and I. D. Abrahams 1994 *Journal of Sound and Vibration* **174**, 353–377. The radiation of sound waves from a lightly loaded finite elastic shell, II: non-linear shell resonances.
4. J. H. Ginsberg 1975 *Journal of Sound and Vibration* **40**, 359–379. Multi-dimensional non-linear acoustic wave propagation, Part II: the non-linear interaction of an acoustic field and plate under harmonic excitation.
5. A. H. Nayfeh and S. G. Kelly 1978 *Journal of Sound and Vibration* **60**, 371–377 Non-linear interactions of acoustic fields with plates under harmonic excitations.
6. Lord Rayleigh 1896 *The Theory of Sound* (2 vols). New York: Dover Publications, second edition, 1945 re-issue.
7. S. V. Sorokin and S. G. Kadyrov 1999 *Journal of Sound and Vibration* **222**, 425–451. Modelling of non-linear oscillations of elastic structures in heavy fluid loading conditions.
8. F. G. Leppington 1976 *Quaterly Journal of Mechanics and Applied Mathematics* **29**, 527–546. Scattering of sound waves by finite membranes and plates near resonance.
9. D. G. Crighton and D. Innes 1984 *Philosophical Transactions of the Royal Society of London* **A312**, 295–341. The modes, resonances and forced response of elastic structures under heavy fluid loading.
10. V. A. Golovanov, A. L. Popov and G. I. Chernyshov 1986 *Prikladnaya Matematika i Mekhanika* **50**, 436–442. Oscillations of a bounded plate in a liquid (in Russian).
11. S. V. Sorokin 1993 *Journal of Applied Mathematics and Mechanics* **57**, 105–115. Asymptotic analysis and numerical solution of the two-level boundary equations of a plane problem of stationary hydro-elasticity.
12. M. C. Junger and D. Feit 1972 *Sound, Structures, and their Interactions*. Cambridge, MA: MIT Press
13. A. S. Vol'mir 1956 *Theory of Flexible Plates and Shells*. Moscow: Fizmatgiz (in Russian).
14. A. H. Nayfeh and D. T. Mook 1979 *Non-linear Oscillations*. New York: John Wiley and Sons.
15. J. J. Thomsen 1997 *Vibrations and Stability: Order and Chaos*. London: McGraw-Hill.
16. S. V. Sorokin 1995 *Journal of Sound and Vibration* **184**, 213–228 Analysis of structural–acoustic coupling problems by a two-level boundary equations method. Part 2
17. S. Wolfram 1991 *Mathematica: A System for doing Mathematics by Computer*. Reading, MA: Addison-Wesley Publishing Co.